3. X, Y are iid N(0,1), then for any t in R,

P(X<=t|X+Y>0)= P(X<=t, X+Y>0) /P(X+Y>0)

For the numerator,

P(X<=t, X+Y>0)= P(X<=t, Y>-X)

=\int(-\infty, t) [\phi(t) \int(-x,\infty) \phi(y) dy]dx

=\int(-\infty, t) [\phi(t) \int(-\infty, x) \phi(y) dy]dx

=\int(-\infty, t) [\phi(t) \Phi(x)] dx, where \phi() = [\Phi()]’.

=1/2 \Phi^2(x) | (-\infty, t) = 1/2 \Phi^2(t).

For the denominator, since X+Y are N(0,2), P(X+Y>0)=1/2

Then P(X<=t|X+Y>0) =\Phi^2(t). where \Phi() is the CDF of standard normal distribution.

6. There are N! possible permutation of the N persons 1,2,…,N with equal probability, hence each permutation has probability 1/N!. Denote Xi as the hat that the ith person gets.

PN (X1=1, X2=2, …, XN-2=N-2, XN-1=N-1, XN=N) = 1/N!

PN (X1=1, X2=2, …, XN-2=N-2 ,XN-1=N-1) =1/N!

PN (X1=1, X2=2, …, XN-2=N-2) = 2!/N!

……

PN (X1=1) = (N-1)!/N!

1. E(Y) = E (sum(i=1 to N) 1{Xi=i} ) = N\*P(Xi=i) =N\* (N-1)!/N!=1
2. E(Y^2)= E(sum(i=1 to N) 1{Xi=i} )^2

= sum(i=1 to N) 1{Xi=i} + 2\*sum(1<=i<j<=N) 1{Xi=i, Xj=j}

= E(Y) +2 \* C(N,2) \* P(Xi=i, Xj=j) =1+N(N-1) \* (N-2)!/N!=1+1=2

Then Var(Y) =E(Y^2) –[E(Y)]^2 =1.

Since RN = Y(N)+RN-Y(N)

E(RN) = E(Y(N)) +E(RN-Y(N)).

E(RN) = 1+ P(Y(N)=0) \* E(RN) + P(Y(N)=1) \* E(RN-1) +…… +P(Y(N)=N-1) \* E(R1)

We want to get P(Y(N)=k) ==denoted as== PN(Y=k).

PN (Y=0) = 1 - PN (Y>=1).

Since PN (Y>=1) = PN ( {X1=1} U {X2=2} U {X3=3} U…U {XN-1=N-1}… U {XN=N} )

= Sum(i=1 to N) PN ({Xi=si}) –Sum(1<=i<j<=n) PN ({Xi=i}\*{Xj=j}) +Sum(1<=i<j<k<=n)PN ({Xi=i}\*{Xj=j}\*{Xk=k})-… +(-1)^N Sum(1<=i1<i2<…<iN-1<=N) PN ({Xi1=i1}\*{Xi2=i2}\*….\*{XiN-1=iN-1})

+(-1)^(N+1) Sum(1<=i1<i2<…<iN<=N) PN ({Xi1=i1}\*{Xi2=i2}\*….\*{XiN-1=iN-1}\*{XiN=iN})

=C(N,1)\*PN (X1=1)-C(N,2)\*PN (X1=1,X2=2)+C(N,3)\*PN (X1=1,X2=2,X3=3)-… +(-1)^N\*C(N,N-1)\*PN (X1=1, X2=2, …, XN-2=N-2 ,XN-1=N-1) +(-1)^(N+1)\* C(N,N)\*PN (X1=1, X2=2, …, XN-2=N-2, XN-1=N-1, XN=N)

=C(N,1)\* (N-1)!/N! –C(N,2)\* (N-2)!/N!+C(N,3)\* (N-3)!/N! -…+(-1)^N\*C(N,N-1)\*1!/N! +(-1)^(N+1)\* C(N,N)\*0!/N!

=1/1!-1/2!+1/3!-…. +(-1)^N\*1/(N-1)! +(-1)^(N+1)\*1/N!

We have PN (Y=0) = 1 - PN (Y>=1) = 1/2!-1/3!+…. +(-1)^(N-1)\*1/(N-1)! +(-1)^N\*1/N!

Consider Y=k (1<=k<=N-2), which could be thought of as the first k persons out of N can take their own hats, while for the other N-k, none can get his own hat.

PN (Y=k) = 1/k! \* PN-k (Y=0) =1/k! \*[ 1/2!-1/3!+…. +(-1)^(N-k)\*1/(N-k)! ]

For k>=N-1, we can easily get PN (Y=N-1)=0, P(Y=N)=1/N!

So E(RN) = 1+ (1/2!-1/3!+…. +(-1)^(N-1)\*1/(N-1)! +(-1)^N\*1/N!) \* E(RN)

+1/1!\*(1/2!-1/3!+…. +(-1)^(N-1)\*1/(N-1)!) \* E(RN-1)

+1/2!\*(1/2!-1/3!+…. +(-1)^(N-2)\*1/(N-2)!) \* E(RN-2)

+……

+1/k!\*(1/2!-1/3!+…. +(-1)^(N-k)\*1/(N-k)!) \* E(RN-k)

+……

+1/(N-3)!\*(1/2!-1/3!) \* E(R3)

+1/(N-2)!\*(1/2!) \* E(R2)

Since E(R1)=1,

E(R2)=1+1/2!\*E(R2) => E(R2)=2.

E(R3)=1+(1/2!-1/3!)\*E(R3)+1/2!\*E(R2) => E(R3)=3.

We try mathematical induction method. Assume E(Rk)=k for all k<=m, then

E(Rm+1)= 1+(1/2!-1/3!+…. +(-1)^m\*1/m! +(-1)^(m+1)\*1/(m+1)!) \* E(Rm+1)

+1/1!\*(1/2!-1/3!+…. +(-1)^m\*1/m!) \* m

+1/2!\*(1/2!-1/3!+…. +(-1)^(m-1)\*1/(m-1)!) \* (m-1)

+……

+1/k!\*(1/2!-1/3!+…. +(-1)^(m+1-k)\*1/(m+1-k)!) \* (m+1-k)

+……

+1/(m+1-3)!\*(1/2!-1/3!) \* 3

+1/(m+1-2)!\*(1/2!) \* 2

Reorganize the equation, and by calculation, we get E(Rm+1) =m+1.

So E(Rk)=k for all k<=m+1.

Then by mathematical induction method, we get E(RN)=N for all N>=1.

1. Since SN = N+SN-Y(N)

E(SN) = N+E(SN-Y(N)).

E(SN) = N+ P(Y(N)=0) \* E(SN) + P(Y(N)=1) \* E(SN-1) +…… +P(Y(N)=N-1) \* E(S1)

Note that E[ SN-N(N-1)/2 ] – E[ SN-Y(N)- (N-Y(N)) (N-Y(N)-1)/2 ]

=E[ SN- SN-Y(N) ] –N(N-1)/2 + N(N-1)/2 + ½ \* E[Y^2(N) -2N\*Y(N) +Y(N) ]

=N+1/2\* [1-2N+1]=1

So E[ SN-N(N-1)/2 ] = 1+ E[ SN-Y(N)- (N-Y(N)) (N-Y(N)-1)/2 ].

So E[ SN-N(N-1)/2 ] has the same property as that of E[RN].

Then E[ SN-N(N-1)/2 ] = E[RN]=N.

So E[ SN ]= N(N-1)/2+N = N(N+1)/2.

1. The expected number of false selections for each of the N persons

1/N\* E(SN)-1 = 1/N\* N(N+1)/2 – 1 = (N-1)/2.

Thanks to example 1.3(a) and 1.5(e) in the book “Stochastic Processes” by S.M. Ross.